Journal of Engineering Mathematics **47:** 61–82, 2003.

Analysis and simulations of vibrations of a beam with a slider

Y. DUMONT[1], K. L. KUTTLER[2] and M. SHILLOR[3]
[1]*IREMIA- Université de La Réunion, Saint-Denis, Ile de La Réunion, France*
[2]*Department of Mathematics, Brigham Young University, Provo, Utah, U.S.A.*
[3]*Department of Mathematics and Statistics, Oakland University, Rochester, MI 48309, U.S.A.,*
e-mail: shillor@oakland.edu

Received 22 April 2002; accepted in revised form 26 February 2003

Abstract. A model for vibrations of a beam with a slider is derived, analysed and numerically simulated. It describes a viscoelastic beam that is clamped at one end to a vibrating device, while the other end moves between two stops attached to a slider. The contact is described by the normal compliance or by the Signorini conditions. The existence of weak solutions is established using the theory of set-valued pseudomonotone operators. The model is discretized using fourth-order spatial discretization, the solutions are numerically simulated and their results presented. The dynamics of the vibrations are depicted and so are the noise characteristics of the system.

Key words: dynamic contact, normal compliance, Signorini's condition, slider, viscoelastic beam

1. Introduction

We continue our investigation of the noise and vibration characteristic of contacting one-dimensional structures, conducted in [1] and [2–4]. In [1] the problem of vibrations of a viscoelastic beam between two rigid or deformable stops was modeled and the existence and possible uniqueness of weak solutions established. Numerical simulations of the problem were performed in [2, 3]. The case when the stops are not stationary but attached to another beam, which is a simple setting for investigating vibrations transmissions across a joint, was investigated and numerically simulated in [4]. Here, we consider a different setting where the stops are attached to a slider which moves on a rail as a result of the contact forces. Thus, the position of the stops is unknown, and forms a part of the problem. We model the process, establish the existence of its weak solution and simulate the behavior of the solutions numerically. The model includes hysteresis via the motion of the slider, and is related to the so-called 'play model,' see, *e.g.*, [5] and references therein.

There is considerable interest in the automotive industry, as well as in other industries, in dynamic vibrations of mechanical systems. In addition to the importance of accurate prediction of these vibrations and their effects on the mechanical behavior and reliability of the system, it is important to understand and control the generated noise. Indeed, appreciable effort has been made in the design of automotive components to reduce unwanted or disturbing noise. A portion of the unwanted noise is generated by the dynamic contact of parts and components when periodically forced. When the mounting of the components on the car engine is not perfect, the motion in the clearances leads to dynamic contact, which in turn may generate unwanted noise.

The main tools used in the investigation of noise and vibration (NVH) are modal analysis and energy density distribution methods. However, these work well only for simple geometries and settings, and do not take contact into account. Often, in the study and simulations of vibrations of elastic bodies and structures, random or stochastic input is assumed – the source of which is usually unspecified. However, nonlinear deterministic systems can exhibit behavior which is essentially unpredictable; namely, they can behave chaotically. These chaotic vibrations may cause certain noise characteristics which need to be well understood before any systematic noise control can be designed, or a proper component setting worked out. Moreover, these chaotic oscillations may remove the need for any stochastic input into the models.

In this work we consider a one-dimensional problem which describes a viscoelastic or elastic beam that is clamped at one end to an oscillating, device while the other end vibrates between two rigid or flexible stops. Such a setting was considered by Moon and Shaw [6] (see also [7, Chapter 4] and references therein) where the mathematical problem was considerably simplified by expanding the solutions in terms of the beam's eigenfunctions and using only the first ones. In this way the problem was reduced to a nonlinear ordinary differential equation. They showed that even such simplified approximation can exhibit complicated types of behavior under periodic forcing. Indeed, it can oscillate periodically, quasiperiodically or chaotically. Clearly, since we consider the full problem, we expect it to show even more varied types of behavior. The novelty here, as compared to [1–4], is the fact that the stops are mounted on a slider, the motion of which is a part of the problem.

A related result for elastic rods can be found in Schatzman [8], and a simplified model for vibrations of a rod in [9] and in [10].

The model is described in Section 2. We choose the material constitutive law to be either elastic or viscoelastic of the Kelvin-Voigt type. To model the contact we use either the Signorini nonpenetration condition, which models two perfectly rigid stops (see, *e.g.*, , [12] or [13]), or the so-called 'normal compliance' condition (see, *e.g.*, [12–17] and references therein), which describes flexible stops with nonlinear spring–like reaction. The classical model for the problem with flexible stops is presented, its variational formulation derived, and the existence results summarized in Theorems 2.2. The variational formulation of the problem with the Signorini condition is presented too. The proof of Theorem 2.2, the existence and uniqueness for the model with normal compliance is given in the Appendix. It is based on the result of [18, 19] for abstract degenerate evolution equations.

In Section 3 we present an algorithm for numerical simulations of the problem. It is based on finite difference discretization of the problem and iterations. Results of numerical simulations are depicted in Section 4. We conclude the paper in Section 5 with a summary and describe some related open problems.

2. Model formulation and results

We present the physical setting of the process, the classical model, its variational formulation and state our main results.

A vertical beam is clamped at its top to a device that may shake or oscillate. Its lower end is constrained to move, horizontally, between two obstacles – the stops – which are mounted on a slider that is attached to a horizontal rail. The physical setting is depicted in Figure 1

The beam's (stress free) reference configuration coincides with the interval $0 \le y \le 1$, and the contacting end is located at $y = 0$. The slider's center of mass is at $x = x(t)$ and the positions of the stops are $x(t) - g_1$ and $x(t) + g_2$ (for $0 < g_1, g_2$). We set $\Omega_T = (0, 1) \times (0, T)$,

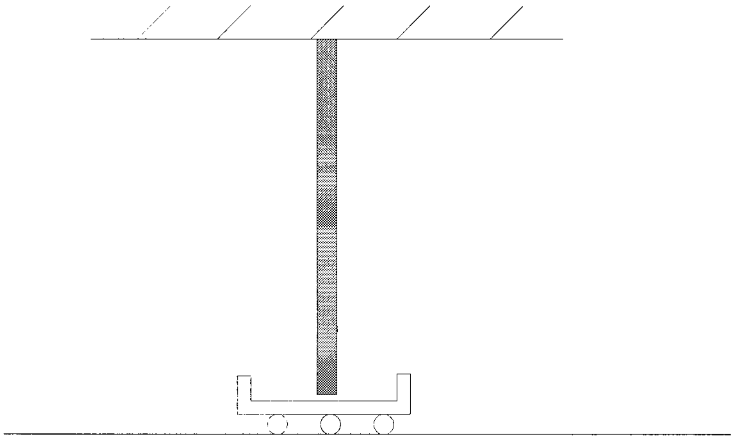

Figure 1. The beam and the slider.

for $T > 0$. Let $u = u(y, t),\ (y, t) \in \Omega_T,$ represent the horizontal displacement of the beam, and $\sigma = \sigma(y, t)$ its shear stress. The equation of motion of the beam, in a nondimensional form, is

$$u_{tt} - \sigma_y = f \qquad \text{in } \Omega_T, \tag{2.1}$$

where $f = f(y, t)$ denotes the density (per unit length) of applied horizontal forces (which we include for the sake of generality), and the subscripts y and t indicate partial derivatives. The material is assumed to be viscoelastic, with constitutive law

$$-\sigma(y, t) = a^2 u_{yyy}(y, t) + da^2 u_{tyyy}(y, t). \tag{2.2}$$

Here, $a^2 = EI/A\rho$, where E is the modulus of elasticity, ρ is the material density, A is the cross section area, I is the second area moment of the cross section, and d is a nonnegative viscosity coefficient. When $d = 0$ the beam is elastic.

Let $\tau = \tau(t) = \sigma(0, t)$ be the shear stress at the free end of the beam, and assume that the slider's mass is m. Then, the equation of motion of the slider obeys Newton's law,

$$mx'' = -\tau. \tag{2.3}$$

Here and below, a prime over a variable indicates a time derivative. The initial conditions are

$$u(y, 0) = u_0(y), \qquad u_t(y, 0) = v_0(y), \tag{2.4}$$

for $0 \le y \le 1$, where u_0 and v_0 represent the beam's initial deflection and velocity, respectively, and

$$x(0) = x_0, \qquad x'(0) = v_0^*, \tag{2.5}$$

where x_0 and v_0^* are the slider's initial position and velocity, respectively. Below, we always assume the compatibility condition $u_0(0) \in [x_0 - g_1, x_0 + g_2]$. The beam is rigidly attached at its top, thus,

$$u(1, t) = \phi(t) \qquad \text{and} \qquad u_y(1, t) = 0, \tag{2.6}$$

where $\phi = \phi(t)$ represents the motion of the supporting device.

At the contacting end we may use either the Signorini condition (see, *e.g.*, [11–13]) or the normal compliance condition (see, *e.g.*, [13, 15–17, 20]).

In this work we use the so-called *normal compliance* condition, which assumes that the stops are flexible. Their one-sided restoring force is assumed to be proportional to their (only outward) deflections from their rest positions,

$$\tau(t) = -\kappa \left[(u(0,t) - x(t) - g_2)_+ - (x(t) - g_1 - u(0,t))_+ \right], \tag{2.7}$$

where κ is the stops' stiffness coefficient, and $(f)_+ = \max\{f, 0\}$ is the positive part of f. The stops behave as one-sided nonlinear springs.

For the sake of completeness, we also describe the Signorini condition since it is often used in publications and applications. The Signorini condition models the idealized case of completely rigid stops, and leads to considerable mathematical difficulties, see, *e.g.*, [1]. Since the stops are rigid, the displacement $u(0,t)$ of the contacting end is allowed only between the stops,

$$x(t) - g_1 \leq u(0,t) \leq x(t) + g_2, \tag{2.8}$$

and then, either the end is free and $\tau(t) = 0$; or it is in contact and the stress is opposite to the displacement,

$$\tau(t) \leq 0 \quad \text{if} \quad u(0,t) = x(t) + g_2; \qquad \tau(t) \geq 0 \quad \text{if} \quad u(0,t) = x(t) - g_1.$$

Moreover, to ensure that only one of the cases takes place it is required that

$$\tau(x + g_2 - u)_+(u - x - g_1)_+ = 0.$$

The Signorini condition may be restated as follows. Let $\chi = \chi(x;r) = \chi_{[x-g_1,x+g_2]}(r)$ be the indicator function of the interval $[x-g_1, x+g_2]$, *i.e.*, $\chi(x;r) = 0$ when $r \in [x-g_1, x+g_2]$ and $\chi(x;r) = +\infty$ otherwise. Then, Signorini's condition, in addition to (2.8), states that the stress satisfies

$$-\tau(t) \in \partial\chi(x(t); u(0,t)), \tag{2.9}$$

for $0 \leq t \leq T$, where $\partial\chi$ is the subdifferential of χ with respect to the second variable, *i.e.*,

$$\partial\chi(x;r) = \begin{cases} [0, +\infty) & r = x + g_2, \\ 0 & x - g_1 < r < x + g_2, \\ (-\infty, 0] & r = x - g_1. \end{cases}$$

This condition is elegant, and easy to write and program, but leads to severe mathematical difficulties which may reflect the fact that in dynamic contact or impact problems the notion of a rigid body may be not very useful. Indeed, any weak solution in this case is not sufficiently regular for the shear stress τ to be well defined as a function, and so, condition (2.3) does not make sense, classically.

Formally, we obtain (2.8) and (2.9) from (2.7) in the limit $\kappa \to \infty$, *i.e.*, when the stops' stiffnes becomes infinite, and therefore, we may consider (2.7) as regularization of the Signorini condition. Alternatively, as we do here, we may consider the Signorini condition as an idealization of the more realistic condition (2.7).

We use condition (2.7) in the next section and in the numerical simulations. In this work it is assumed piecewise linear, although we could have chosen a nonlinear expression, such as a power law, as well. It might be of interest to find the form of (2.7) experimentally, by using parameter identification optimization.

Finally, we assume that no moments act on the contacting end,

$$u_{yy}(0,t) + du_{yyt}(0,t) = 0. \tag{2.10}$$

We note that the motion of the slider introduces additional complexity to the problem, compared to the one in [1, 3], and the problem is comparable to the one in [4].

The classical statement of the problem of *vibrations of a beam between two reactive stops attached to a slider* is: Find a pair of functions $\{u, x\}$ such that (2.1)–(2.7) and (2.10) hold.

The problem with the Signorini condition is: Find a pair of functions $\{u, x\}$ such that (2.1)–(2.6), (2.8)–(2.10) hold.

Both contact conditions impose on the solutions *regularity ceilings* which, generally, preclude the existence of classical solutions for the problems. This is related, in part, to the possible discontinuity of the velocity upon impact. Thus, it is natural, actually necessary, to consider weak or variational formulation of the problems. Toward that end we introduce the following spaces and notation. For definitions of any unexplained terms we refer the reader to [21, Chapters 2–4] or [22, Chapter 1].

Let $H = L^2(0,1)$ and $V = \{w \in H^2(0,1) : w(1) = w'(1) = 0\}$, then

$$V \subseteq H = H' \subseteq V',$$

where V' is the topological dual of V, so $\{V, H, V'\}$ is a Gelfand triplet. We denote by $(\cdot,\cdot)$ the inner product in H and by $\langle\cdot,\cdot\rangle$ the duality pairing between V and V'. The associated norm on H is denoted by $|\cdot|_H$ and the one on V by $||\cdot||_V$. We note that the following is an equivalent norm for V,

$$||u|| \equiv \left(\int_0^1 \left|u_{yy}\right|^2 \mathrm{d}y\right)^{1/2},$$

and we shall use this whenever convenient. Also, we use the notation $\mathcal{V} = L^2(0,T;V)$, and then $\mathcal{V}' = L^2(0,T;V')$.

Next, to have zero boundary condition at $y = 1$ we change the dependent variable to $u(y,t) = \bar{u}(y,t) + \eta(y)\phi(t)$, where η is a fixed smooth function defined on $[0,1]$ such that $\eta(0) = \eta'(0) = \eta''(0) = \eta'''(0) = \eta'(1) = 0$ and $\eta(1) = 1$. These requirements guarantee that the other boundary conditions do not change, and we find easily that

$$\eta\left(y\right) = y^4\left(5 - 4y\right),$$

satisfies all of the conditions. Then the forcing function f changes to

$$\bar{f} = f - \eta\phi'' - a^2\eta_{yyyy}\phi - da^2\eta_{yyyy}\phi'.$$

Below we omit the bar and use the transformed $\bar{u}$ and $\bar{f}$.

We formulate the problem with normal compliance first. Then we discuss briefly the one with Signorini's condition.

To derive the variational formulation, let $\phi \in C_c^\infty(0,T)$ be a test function and let $w \in V$. We multiply (2.1) by ϕw and integrate, thus,

$$-\int_0^T \int_0^1 u_t \phi' w \,\mathrm{d}y\mathrm{d}t - \int_0^T \sigma \phi w|_0^1 \,\mathrm{d}t + \int_0^T \int_0^1 \sigma w_y \phi \,\mathrm{d}y\mathrm{d}t = \int_0^T \int_0^1 f w \phi \mathrm{d}y\mathrm{d}t,$$

and from the normal compliance condition, (2.7), we obtain

$$\begin{aligned} &-\int_0^T \int_0^1 u_t \phi' w \mathrm{d}y\mathrm{d}t + \int_0^T \int_0^1 \sigma w_y \phi \mathrm{d}y\mathrm{d}t + \kappa \int_0^T \big[(u(0,t) - x(t) - g_2)_+ \\ &- (x(t) - g_1 - u\,(0,t))_+\big]\,\phi\,(t)\,w\,(0)\,\mathrm{d}t = \textstyle\int_0^T \int_0^1 f w \phi \mathrm{d}y\mathrm{d}t. \end{aligned} \tag{2.11}$$

Now, it follows from (2.2) that

$$\begin{aligned} \int_0^T\!\!\int_0^1 \sigma w_y \phi \mathrm{d}y\mathrm{d}t &= \int_0^T \int_0^1 \big(-a^2 u_{yyy}(y,t) - da^2 u_{yyyt}(y,t)\big)\, w_y \phi \mathrm{d}y\mathrm{d}t \\ = &\int_0^T \big(-a^2 u_{yy}(y,t) - da^2 u_{yyt}(y,t)\big)\, w_y \phi|_0^1 \mathrm{d}t + \int_0^T\!\!\int_0^1 a^2 u_{yy}(y,t) + da^2 u_{yyt}(y,t) w_{yy} \phi \mathrm{d}y\mathrm{d}t. \end{aligned} \tag{2.12}$$

Then (2.10) implies that this term reduces to

$$\int_0^T \int_0^1 \big(a^2 u_{yy}(y,t) + da^2 u_{yyt}(y,t)\big)\, w_{yy} \phi \mathrm{d}y\mathrm{d}t. \tag{2.13}$$

We now present the variational problem in terms of the following operators. First, we define the nonlinear operator $P : [0,T] \times H^2\,(0,T) \times \mathcal{V} \to \mathcal{V}'$ by

$$\langle P\,(t,x,u)\,,w\rangle \equiv \int_0^T \kappa\,\big((\gamma_0 u - x(t) - g_2)_+ - (x(t) - g_1 - \gamma_0 u)_+\big)\,\gamma_0 w \,\mathrm{d}t, \tag{2.14}$$

and we denote by $P\,(x,u)$ the operator mapping $H^2\,(0,T) \times \mathcal{V} \to \mathcal{V}'$ which is given by $P\,(x,u)\,(t) \equiv P\,(t, x\,(t)\,, u\,(t))$. Moreover, we define the linear operator $C : V \to V'$ by

$$\langle Cu, w\rangle \equiv a^2 \int_0^1 u_{yy} w_{yy} \mathrm{d}y. \tag{2.15}$$

The variational formulation of the problem of *vibrations of a viscoelastic beam between two reactive stops attached to a slider* is as follows:

Definition 2.1. A pair of functions $\{u, x\}$ such that $u, v = u' \in \mathcal{V},\ \ v' \in \mathcal{V}'$ and $x \in H^2(0,T)$ is said to be a weak solution of (2.1)–(2.7) and (2.10), provided that

$$u\,(t) = u_0 + \int_0^t v\,(s)\,\mathrm{d}s, \tag{2.16}$$

for $u_0 \in V$, and the evolution equations,

$$v' + Cu + dCv + P\,(x,u) = f, \tag{2.17}$$

$$x'' + \frac{\kappa}{m}\big((u(0,t) - x(t) - g_2)_+ - (x(t) - g_1 - u\,(0,t))_+\big) = 0, \tag{2.18}$$

and initial conditions,

$$v\,(0) = v_0 \in H,\ x\,(0) = x_0,\ x'\,(0) = v_0^* \tag{2.19}$$

are satisfied.

We have the following existence and uniqueness result for the problem.

Theorem 2.2. Let $d \geq 0$, and assume that $f \in L^2(0, T; H)$, $\phi \in H^2(0, T)$, $u_0 \in V$, $v_0 \in H$. If $d > 0$, problem (2.16)–(2.19) has a unique weak solution for each $T < \infty$, and the solution satisfies

$$u \in L^\infty(0, T; V), \ \ v \in \mathcal{V} \cap L^\infty(0, T; H), \ \ v' \in \mathcal{V}'.$$

When $d = 0$, and initially $x_0 - g \leq u_0(0) \leq x_0 + g$, then the problem has a solution which satisfies

$$u \in L^\infty(0, T; V), \ \ v \in L^\infty(0, T; H), \ \ v' \in \mathcal{V}'.$$

The proof of the theorem is given in the Appendix, is quite technical and involves set-valued pseudomonotone operators, approximate problems and a priori estimates.

We conclude that problem (2.1)–(2.7) and (2.10) has a weak solution for each $T < \infty$. When the viscosity is retained, $0 < d$, the weak solution is unique.

We turn briefly to the Signorini conditions (2.8) and (2.9). To state a weak formulation for the problem including it, we introduce the set of admissible displacements

$$K_x = \{w \in V : x - g_1 \leq w(0) \leq x + g_2\},$$

which is a convex set and $x = x(t)$. Then, a pair of functions $\{u, x\}$ such that $u \in \mathcal{V}$ and $x \in H^2(0, T)$ is said to be a weak solution of (2.1)–(2.6) and (2.8)–(2.10) provided that: $u' \in \mathcal{V}$, $u(t, \cdot) \in K_{x(t)}$, a.e. t, $x(0) = x_0, x'(0) = v_0^*$ and $u(\cdot, 0) = u_0 \in K$, and for each $w \in \mathcal{V}$, such that $w(\cdot, t) \in K_{x(t)}$, a.e. t, $w' \in L^2(0, T; H)$ and $w(\cdot, T) = u(\cdot, T)$,

$$-\int_0^T (u', \ w' - u')\,\mathrm{d}t + a^2 \int_0^T (u_{xx}, \ w_{xx} - u_{xx})\mathrm{d}t + d \int_0^T (u'_{xx}, \ w_{xx} - u_{xx})\mathrm{d}t$$

$$\geq -(v_0, w(0) - u_0) + \int_0^T (f, \ w - u)\mathrm{d}t, \tag{2.20}$$

and

$$mx'' = -\tau(t) \in \partial\chi(x(t); u(0, t)), \tag{2.21}$$

which holds in the sense of distributions, and $\tau(t) = \sigma(0, t)$.

The formulation is a variational inequality, and condition (2.23) poses considerable mathematical difficulties. Indeed, $\tau(t) = \sigma(0, t)$ is only a distribution. We do not consider the Signorini condition further in this work. However, we remark that we can pass to the limit $\kappa \to \infty$ in the problem with $d > 0$ and obtain a weak solution for this problem, similarly to what has been done in [1].

Remark 2.3. The assumption that the contact between the beam and the stops takes place only at the beam's tip is an idealization of the real situation. The physical contact occurs over an interval $0 \leq y \leq \epsilon$, where a boundary layer exists. The problem above might be obtained by using asymptotic expansion in the limit $\epsilon \to 0$. The problem with ϵ has interest by and of itself, and will be considered in the future. We just note that in the limit problem contact enters via the boundary condition at the tip, while for the problem with ϵ the contact enters via the reaction force in the beam equation, see, *e.g.*, [23].

3. Numerical algorithm

We turn to a numerical algorithm for the problem. For the sake of simplicity, in this and the next section it is assumed that the slider is symmetric, *i.e.*, $g_1 = g_2 = g$. We write (2.1)–(2.7) and (2.10), as

$$u_{tt}(y,t) + a^2 u_{yyyy}(y,t) + da^2 u_{tyyyy}(y,t) = f(y,t),$$
$$u(y,0) = u_0(y) - \phi(0)\eta(y), \quad u_t(y,0) = u_0(y) - \phi'(0)\eta(y),$$
$$u(1,t) = 0, \quad u_y(1,t) = 0, \quad u_{yy}(0,t) + du_{yyt}(0,t) = 0,$$
$$\sigma(0,t) = -\kappa\left[(u(0,t) - x(t) - g)_+ - (x(t) - g - u(0,t))_+\right],$$
$$mx''(t) = -\sigma(0,t), \quad x(0) = x_0, \quad x'(0) = v_0^*.$$

First, we discretize the displacements u of the beam. Let $\Delta t = T/N$ be the time step and let $h = 1/M$ be the space step, then the grid points are given by $(y_i = ih,\ t_n = n\Delta t)$, for $i = 0, \cdots, M$ and $n = 0, \cdots, N$.

3.1. Space discretization

The equation in the interior of the beam can be written as

$$u_{tt}(y,t) = -a^2 u_{yyyy}(y,t) + \bar{d}\,(u_t(y,t))_{yyyy} + f(y,t),$$

where $\bar{d} = a^2 d$.

We discretize the equation with respect to the space variable using a classical fourth order finite difference scheme. We obtain a system of $M-2$ ordinary differential equations ($i = 1, \ldots, M-2$),

$$\begin{aligned}\frac{\mathrm{d}^2 u_i}{\mathrm{d}t^2} = &-\frac{a^2}{h^4}\left(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}\right)\\ &-\frac{\bar{d}}{h^4}\left(\frac{\mathrm{d}u_{i-2}}{\mathrm{d}t} - 4\frac{\mathrm{d}u_{i-1}}{\mathrm{d}t} + 6\frac{\mathrm{d}u_i}{\mathrm{d}t} - 4\frac{\mathrm{d}u_{i+1}}{\mathrm{d}t} + \frac{\mathrm{d}u_{i+2}}{\mathrm{d}t}\right) + f_i,\end{aligned} \tag{3.1}$$

where $u_i = u_i(t) = u(y_i, t)$ and $f_i = f_i(t) = f(x_i, t)$. Then, using the boundary conditions $u(1,t) = u_y(1,t) = 0$, we set $u_{M-1}(t) = u_M(t) = 0$. Moreover, from the assumption $u_y(1,t) = 0$ we deduce

$$\begin{aligned}\frac{\mathrm{d}^2 u_{M-2}}{\mathrm{d}t^2} = &-\frac{a^2}{h^4}\left(7u_{M-1} - 4u_{M-2} + u_{M-3}\right)\\ &-\frac{\bar{d}}{h^4}\left(7\frac{\mathrm{d}u_{M-1}}{\mathrm{d}t} - 4\frac{\mathrm{d}u_{M-2}}{\mathrm{d}t} + \frac{\mathrm{d}u_{M-3}}{\mathrm{d}t}\right) + f_{M-2}.\end{aligned} \tag{3.2}$$

Now, for $i = 0$, recalling that $u_{yyy}(0,t) + \bar{d}u_{yy}(0,t) = -\sigma(0,t)$ and $u_{yy}(0,t) + \bar{d}u_{yy}(0,t) = 0$, and using Taylor's expansion at the virtual points $y = -h$ and $y = -2h$, we obtain

$$\begin{aligned}\frac{\mathrm{d}^2 u_0}{\mathrm{d}t^2} = &-\frac{a^2}{h^4}\left(2u_0 - 4u_1 + 2u_2\right)\\ &-\frac{\bar{d}}{h^4}\left(2\frac{\mathrm{d}u_0}{\mathrm{d}t} - 4\frac{\mathrm{d}u_1}{\mathrm{d}t} + 2\frac{\mathrm{d}u_2}{\mathrm{d}t}\right) - f_0 + \frac{2\sigma_0(t)}{h},\end{aligned} \tag{3.3}$$

where $\sigma_0(t) = \sigma(0,t)$. Next, taking $i = 1$ in (3.1) and using again the assumption that $u_{yy}(0,t) + du_{tyy}(0,t) = 0$, we find

$$\begin{aligned}\frac{\mathrm{d}^2 u_1}{\mathrm{d}t^2} = &-\frac{a^2}{h^4}(u_3 - 4u_2 + 5u_1 - 2u_0)\\ &-\frac{\bar{d}}{h^4}\left(\frac{\mathrm{d}u_3}{\mathrm{d}t} - 4\frac{\mathrm{d}u_2}{\mathrm{d}t} + 5\frac{\mathrm{d}u_1}{\mathrm{d}t} - 2\frac{\mathrm{d}u_0}{\mathrm{d}t}\right) + f_1.\end{aligned} \tag{3.4}$$

Finally, equations (3.1)–(3.4) may be written as the following system of second order ODEs,

$$\mathbf{u}'' = a^2\mathbb{H}\mathbf{u} + \bar{d}\mathbb{H}\mathbf{u}' + F + G. \tag{3.5}$$

Here, $\mathbf{u} = (u_0, \ldots, u_{M-2})^T$, is the vector of nodal displacements of the beam, $\mathbb{H}$ is the $(M-1)\times(M-1)$ matrix

$$\mathbb{H} = \frac{1}{h^4}\begin{pmatrix} -2 & 4 & -2 & 0 & 0 & \ddots & 0\\ 2 & -5 & 4 & -1 & 0 & \ddots & 0\\ -1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & -1\\ \vdots & \ddots & \ddots & -1 & 4 & -6 & 4\\ 0 & \cdots & \cdots & 0 & -1 & 4 & -7 \end{pmatrix},$$

the force vector is given by

$$F^T = F(t)^T = (f_0(t)\ldots f_i(t)\ldots f_{M-2}(t))^T,$$

and the contact shear stress vector is

$$G = \frac{2}{h}\kappa\begin{pmatrix} \big((x(t) - g - u_0(t))_+\big) - (u_0(t) - x(t) - g)_+\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

We next describe the solution method for the system of ODEs (3.5).

3.2. Time integration of the system

A numerical study of the problem with fixed stops has been done in [2]. It has been shown there that the HHT-α family of schemes [24] performed well and produced good results. In particular, these schemes avoid the appearance of spurious high frequency noise by introducing some numerical damping.

The HHT-α method for the system

$$\psi'' = A\psi + B\psi' + f,$$

is constructed as follows. Let (ψ_n, z_n, ζ_n) be an approximation of $\left(\psi(t_n), \psi'(t_n), \psi''(t_n)\right)$, then

$$\begin{cases} \zeta_{n+1} = (1+\alpha)(A\psi_{n+1} + Bz_{n+1}) - \alpha(A\psi_n + Bz_n) + f_{n+1}, \\ \psi_{n+1} = \psi_n + \Delta t z_n + (\Delta t)^2 \left(\beta\zeta_{n+1} + \left(\frac{1}{2} - \beta\right)\zeta_n\right), \\ z_{n+1} = z_n + \Delta t \left(\gamma\zeta_{n+1} + (1-\gamma)\zeta_n\right), \end{cases} \tag{3.6}$$

for $0 \le n \le N-1$. Here α, β and γ are parameters which control the stability, the accuracy and the damping of the numerical solutions. By fixing the values of γ and β, we reduce the HHT family to a one parameter family. In particular, the HHT-α algorithm (see, *e.g.*, [24]) is:

- quadratically convergent if $\gamma = \frac{1}{2} - \alpha$ and $\beta = \frac{1}{4}(1-\alpha)^2$,
- unconditionaly stable for linear problems if $\alpha \in \left[-\frac{1}{3}, 0\right]$.

The parameter α controls the damping of the higher modes in the solution. When $\gamma = \frac{1}{2} - \alpha$, $\beta = \frac{1}{4}(1-\alpha)^2$ and $\alpha = 0$, we recover the well-known Newmark's trapezoidal rule, which yields a second order, non-dissipative and energy preserving algorithm. However, the second order Newmark's scheme doesn't provide satisfactory results because of the presence of high-frequency oscillations. The trapezoid method cannot simultaneously yield quadratic accuracy and dampen the high-frequency effects.

Remark 3.1. First order Newmark's schemes, *i.e.*, $\alpha = 0$ and $\gamma > \frac{1}{2}$, introduce some numerical damping in order to stabilize the integration. But, they also introduce amplitude decay and period elongation which affect the lower modes of the solutions. It was shown in [24] that the HHT method has better dissipation properties than Newmark's.

Remark 3.2. It is possible to rewrite (3.6) in a 'velocity-displacement formulation' when $d > 0$ and in 'displacement' formulation when $d = 0$.

We now discretize the equation of motion of the slider. Let $x^n = x(t_n)$, then (2.3), to order $O((\Delta t)^3)$, is approximated by

$$x^n = 2x^{n-1} - x^{n-2} - \frac{1}{m}(\Delta t)^2 \sigma_0(t_n). \tag{3.7}$$

Here, the force $\sigma_0(t_n)$ is given by

$$\sigma_0(t_n) = \kappa\left(\left(x^{n-1} - g - u_0(t_n)\right)_+ - \left(u_0(t_n) - x^{n-1} - g\right)_+\right). \tag{3.8}$$

We note that $\sigma_0(t_n)$ is computed from the previous position of the slider and the new position of the beam's tip. This is a good approximation as long as the displacement increments are small, which means that the time step must be sufficiently small.

The algorithm for the problem, after initialization and computation of $\mathbf{u}$ and x up to time level $t_{n-1} = (n-1)\Delta t$, proceeds in two steps:

- Solve the equations for the displacements of the beam. Thus, we solve the time discretized system of equations (3.5) for $\mathbf{u}^n$, and then we compute the force $\sigma_0(t_n)$ from (3..8);
- Compute the position of the slider x^n, using (3.7), and then find the new positions of the stops

ω	Δt	m	E	κ
10	10^{-4}	100000	0·2	2×10^6

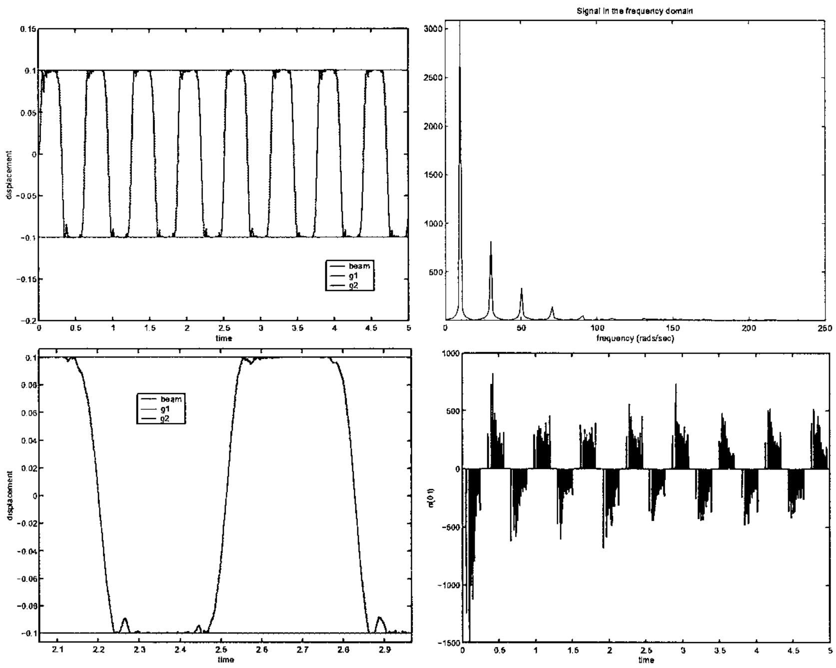

Figure 2 A heavy slider

$$x\,(t_n) - g, \quad x\,(t_n) + g.$$

And then proceed to time level t_{n+1}.

4. Numerical simulations

We now describe simulations performed, by employing the HHT-α scheme, with $\alpha = -0{\cdot}01$. We used $d = 0$ and $d > 0$, various driving frequencies and various masses of the slider. The purpose of the simulations was two-fold: to obtain confidence in the algorithm, and to obtain representative numerical solutions, including their noise.

The vibrations of the supporting device were chosen as

$$\phi(t) = E \sin \omega t,$$

and the values of E and ω are indicated in the tables.

For each of the simulations we present the values of the parameters in a table. Then, we depict the displacement of the beam's end and of the slider or the stops in time, the FFT of the displacement of the beam's tip, a detail of the motion within a short time period, and the contact force $\sigma(0, t)$.

In all the simulation we used the following values of the parameters:

$$a^2 = 300, \qquad h = 0{\cdot}04\ (M = 25), \qquad g = 0{\cdot}1.$$

ω	Δt	m	E	κ
10	10^{-4}	100	0·2	2×10^6

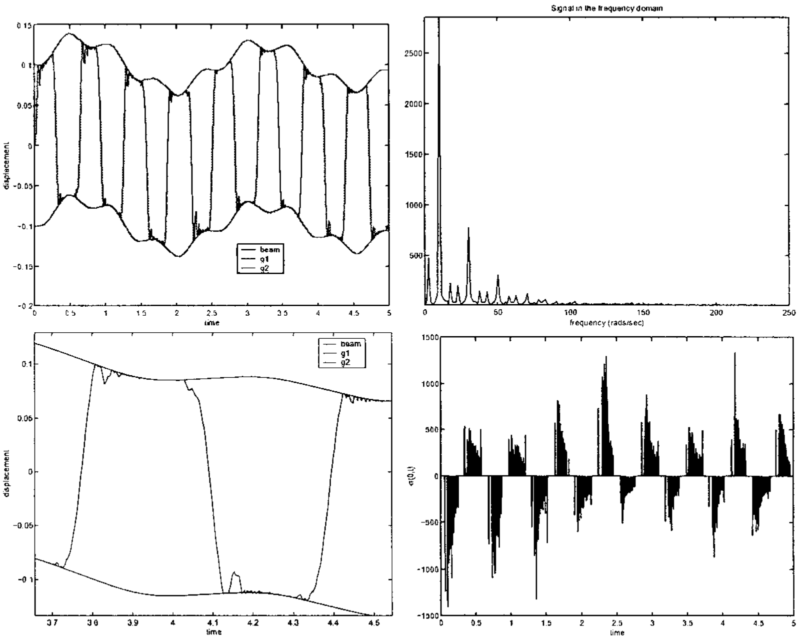

Figure 3 A lighter slider

Moreover, unless indicated otherwise the initial velocity of the slider was $v_0^* = 0$.

We begin by describing simulations of an elastic beam, *i.e.*, when $d = 0$. First we consider a very heavy slider and low frequency ω, Figure 2.

The slider moves very little and the system's behavior is essentially the same as in the case when the stops are rigid [2]. We note that in addtion to the driving frequency $\omega = 10$, few odd multiples of it are also excited. The use of very large normal compliance coefficient κ shows almost no interpenetration, and the system behaves in the same way as with the Signorini condition.

In Figure 3 we depict a lighter slider. More noise frequencies are excited, but with less amplitude. The detail shows that within the periods when the beam's end is mostly in contact with a stop, contact may be lost for brief time intervals due to higher oscillation frequencies of the beam. Also, the contact pressure oscillates rapidly within the contact zones.

Next, we consider a light slider and driving frequency near the beam's first natural frequency. The solution is depicted in Figure 4.

In the next simulation, Figure 5, the support was stationary so that $f = 0$, the slider was very light and had the initial velocity $v_0^* = -1$.

We now suppose again that the support is moving and the slider has an initial velocity $v_0^* = -1$, Figure 6. We note that the system is close to resonance.

We now turn to a simulation with viscosity, and let $d = 0{\cdot}01$. We note, in Figure 7, that the numerical solution is more regular.

ω	Δt	m	E	κ
60	$5{\cdot}10^{-5}$	100	0·2	2×10^7

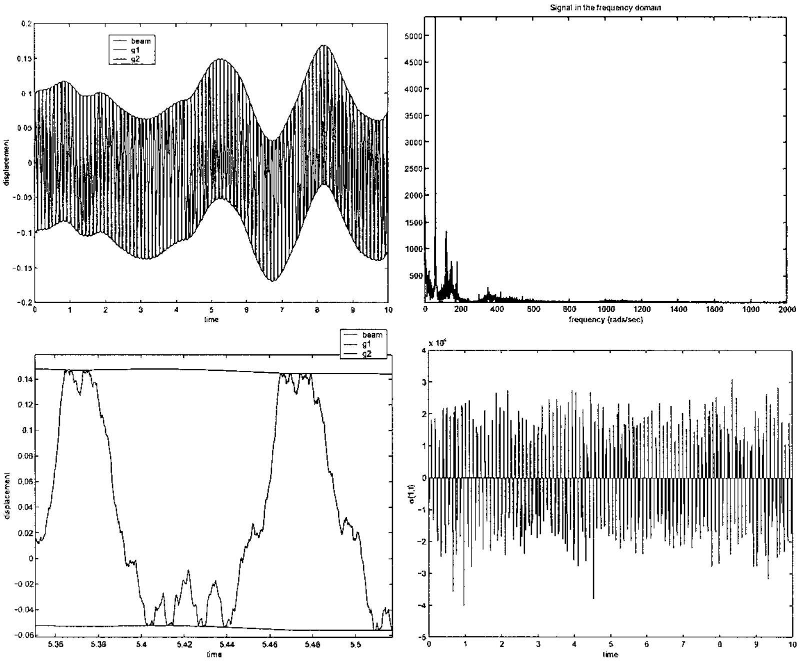

Figure 4 A light slider and driving frequency near resonnance

It is known that impact systems could exhibit some chaotic behavior for a wide variety of parameters. We present a very irregular, ‘possibly’ chaotic behavior, using a very high driving frequency. In Figure 8(b), we show that a wide band of frequencies are excited. We do not know if it is a numerical noise or the system behaves very irregularly. This topic deserves a separate study.

5. Conclusions

We presented a model for the vibrations of a beam between two reactive stops mounted on a slider. The existence of the unique weak solution has been established for the normal compliance contact condition when the beam is viscoelastic and the existence of a weak solution

Δ	g	m	v_0^*	κ
10^{-4}	0·001	10	-1	2×10^6

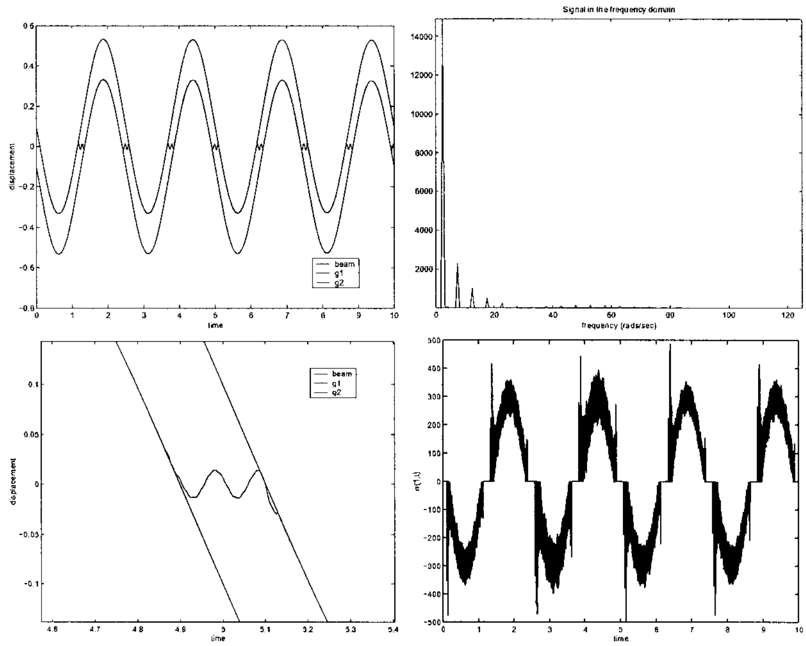

Figure 5 No forcing, initial velocity $v_0^* = -1$

ω	κ	Δt	m	E	v_0^*
10	2×10^6	10^{-4}	10	0·2	-1

when it is elastic. The problem with Signorini's condition, which assumes that the stops are perfectly rigid was left open, deemed of very little applied interest.

An explicit numerical algorithm for the problem was developed. The numerical simulations show that the algorithm performed well and that the system is capable of interesting behavior. The FFT of the motion of the beam's tip indicates that the system may be noisy in certain parameter ranges. Moreover, the solution with high frequency indicates a possible chaotic motion. Figure 6 indicates that there is an interesting resonance in the system that merits further study.

The use of very large normal compliance coefficient κ in the contact stress (2.7) produced numerical solutions without any noticeable interpenetration, and the system behaved in the same way as with the Signorini condition, without the need to implement the latter.

The analysis of the algorithm and the conditions necessary for its convergence are of considerable importance, since they guarantee that the numerical solutions are close to those of the model. Comparison of the numerical solutions with experimental measurements may determine how well the model describes the real system. The determination of what is the behavior of the system and what is a numerical noise are left open at this stage, and are worthy of a careful and detailed study.

We note, in closing, that the slider introduces hysteresis into the problem which has interest by and of itself and should be investigated for its own sake. In this work we did not dwell upon

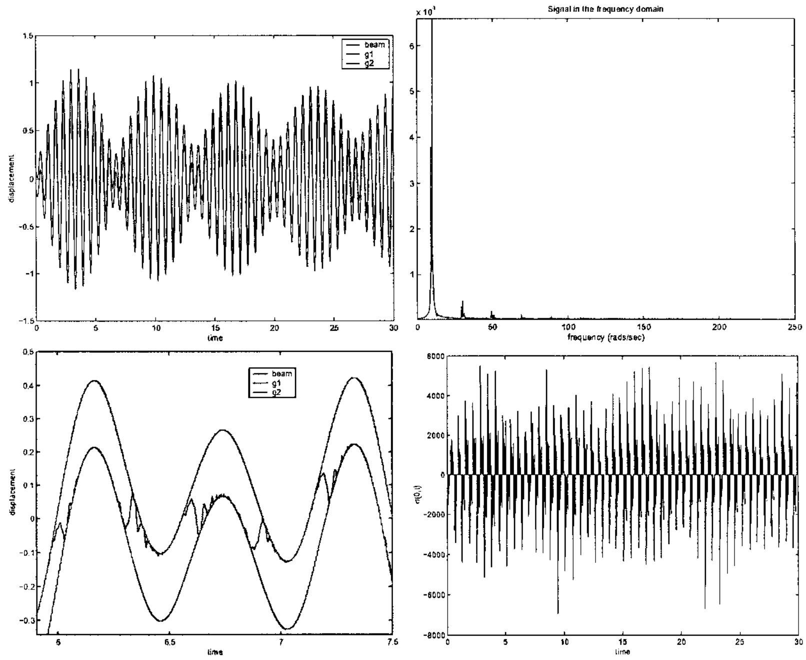

Figure 6 With forcing, initial velocity $v_0^* = -1$

ω	Δt	m	E
10	10^{-4}	100	0·2

this feature of the model. Moreover, taking into account friction between the slider and the rail may be of interest.

Appendix

Proof of Theorem 2.2

In this section we prove Theorem 2.2. First, we introduce the notation

$$R\,(u, x) \equiv \frac{\kappa}{m}\left((u - x - g_2)_+ - (x - g_1 - u)_+\right).$$

Thus, $P\,(x, u) = \gamma_0^* R\,(\gamma_0 u, x)$. We note that R is Lipschitz, indeed,

ω	κ	Δt	m	E	k^2
1000	6×10^7	2×10^{-5}	100	0·2	300

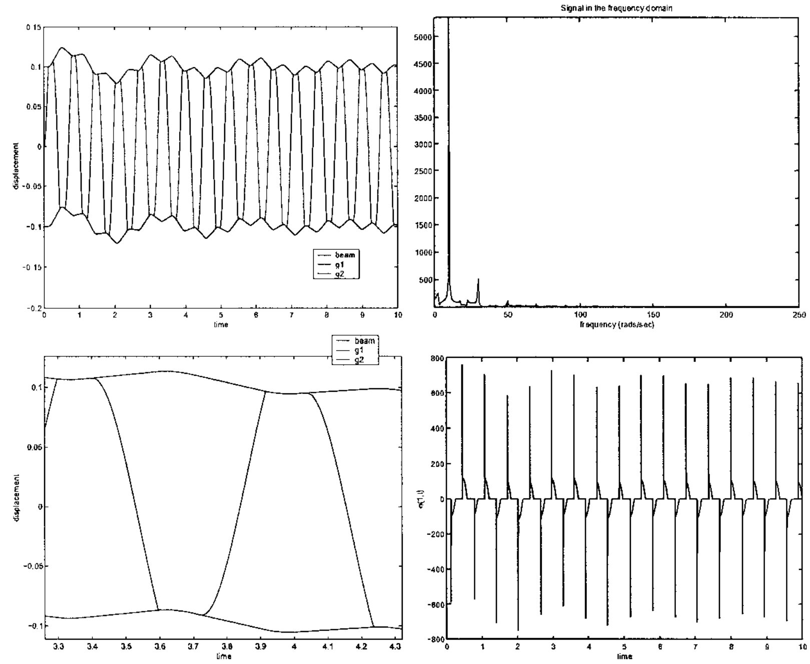

Figure 7 A solution with viscosity

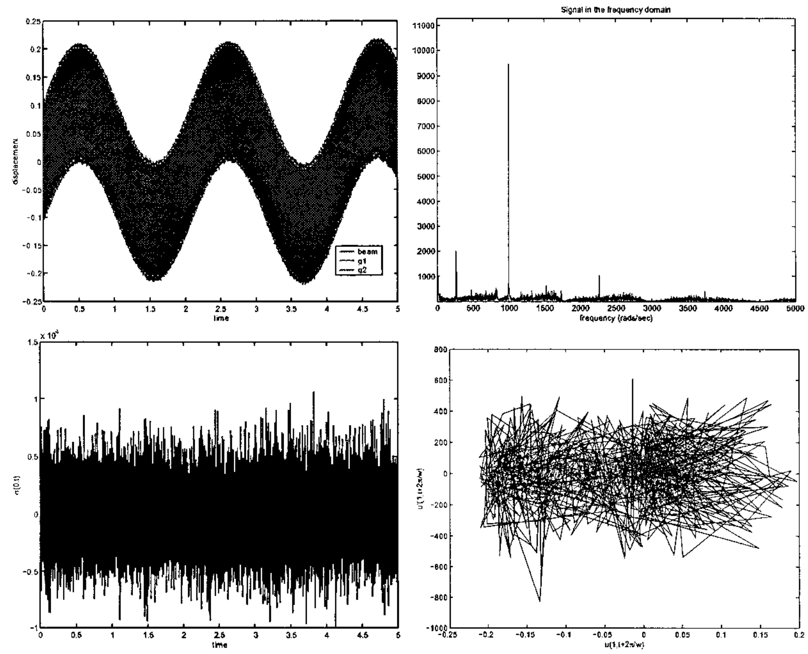

Figure 8 A possible chaotic behavior

$$|R(u_2,x_2) - R(u_1,x_1)| \leq K(m,\kappa)(|x_2 - x_1| + |u_2 - u_1|), \tag{A.1}$$

where $K(m,\kappa)$ is a constant which depends on m and κ. Also, we note that

$$(R(u_1,x_1) - R(u_2,x_2))(u_1 - u_2) = (R(u_1,x_1) - R(u_2,x_1))(u_1 - u_2)$$

$$+ (R(u_2,x_1) - R(u_2,x_2))(u_1 - u_2) \geq (R(u_2,x_1) - R(u_2,x_2))(u_1 - u_2), \tag{A.2}$$

since the function $u \to m^{-1}\kappa\left[(u - x - g_2)_+ - (x - g_1 - u)_+\right]$ is monotone for fixed x, implying that the term on the right-hand side of the first line in (A.2) is nonnegative. Also, by (A.1),

$$|(R(u_2,x_1) - R(u_2,x_2))(u_1 - u_2)| \leq K(m,\kappa)|x_2 - x_1||u_1 - u_2|,$$

and so

$$(R(u_1,x_1) - R(u_2,x_2))(u_1 - u_2) \geq -K(m,\kappa)|x_2 - x_1||u_1 - u_2|. \tag{A.3}$$

Proof of Theorem 2.2: Assume, first, that the beam is viscoelastic, $d > 0$. We fix $x \in C([0,T])$ and consider the following problem,

$$\left(\begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\right)' + \begin{pmatrix} -Cv \\ Cu + dCv + P(x,u) \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \tag{A.4}$$

$$\begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{A.5}$$

Let $F \equiv V \times V$, $G = V \times H$ and let $\mathbb{F} \equiv L^2(0,T;F)$. We define the operators $B : G \to G'$ and $A_x(t) : F \to F'$ by

$$B\begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

$$A_x(t)\begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} -Cv \\ Cu + dCv + P(t,x(t),u) \end{pmatrix}.$$

Next, let

$$l \equiv \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad w_0 \equiv \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \quad \text{and} \quad w \equiv \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then, problem (A.4) and (A.5) may be written as

$$(Bw)' + A_x w = l \quad \text{in } \mathbb{F}, \tag{A.6}$$

$$Bw(0) = Bw_0 \quad \text{in } G', \tag{A.7}$$

which is an implicit evolution problem of the type considered in [18, 19]. Let $W = H^1(0,1)$, so that V embeds compactly into W, and W embeds continuously into $C(0,1)$. Let (w_1,x_1), $(w_2,x_2) \in F \times H^2(0,T)$ be given by $(w_i,x_i) = (u_i,v_i,x_i)$, for $i = 1, 2$, and let $\lambda > 0$, then

$$\langle\left(\lambda B + A_{x_1}\right) w_1 - \left(\lambda B + A_{x_2}\right) w_2,\ w_1 - w_2\rangle$$

$$= \lambda\langle B(w_1 - w_2),\ w_1 - w_2\rangle + \langle A_{x_1} w_1 - A_{x_2} w_2,\ w_1 - w_2\rangle$$

$$= \lambda\Big(\ ||u_1 - u_2||_V^2\ + |v_1 - v_2|_H^2\ \Big) + d\ ||v_1 - v_2||_V^2$$

$$+ \left(\gamma_0^* R\left(\gamma_0 u_1, x_1(t)\right) - \gamma_0^* R\left(\gamma_0 u_2, x_2(t)\right)\right)(v_1 - v_2)\ ,$$

and by (A.3) this satisfies

$$\geq \lambda\ ||u_1 - u_2||_V^2 + \lambda\ |v_1 - v_2|_H^2 + d\ ||v_1 - v_2||_V^2$$

$$-K\ (m, \kappa)\ |\gamma_0 v_1 - \gamma_0 v_2|\ |x_1(t) - x_2(t)|\ . \tag{A.8}$$

Then, using the compactness of the embedding of V into W, along with the inequality

$$||z||_W \leq \epsilon\ ||z||_V + C_\epsilon\ |z|_H\ ,$$

which follows from the compactness of the embedding, we obtain the inequality, in which $C\ (d, m, \kappa)$ depends only on the indicated quantities and is independent of λ,

$$\langle\left(\lambda B + A_{x_1}\right) w_1 - \left(\lambda B + A_{x_2}\right) w_2,\ w_1 - w_2\rangle \geq \lambda\ ||u_1 - u_2||_V^2 + \lambda\ |v_1 - v_2|_H^2$$

$$+ (d/2)\ ||v_1 - v_2||_V^2 - C\ (d, m, \kappa)\ |v_1 - v_2|_H^2 - |x_1(t) - x_2(t)|^2\ . \tag{A.9}$$

We conclude that when λ is sufficiently large, there exists $\delta > 0$ such that

$$\langle\left(\lambda B + A_{x_1}\right) w_1 - \left(\lambda B + A_{x_2}\right) w_2,\ w_1 - w_2\rangle$$

$$\geq \delta\ ||w_1 - w_2||_F^2 - |x_1(t) - x_2(t)|^2 \tag{A.10}$$

In particular, if $x_1 = x_2$, this shows uniqueness of the solution of problem (A.6) and (A.7), thanks to [18].

Now we define a mapping $\Phi : H^2(0, T) \to H^2(0, T)$ as follows. We start with $x \in H^2(0, T)$ and let (u, v) be the solution of problem (A.6) and (A.7) with this x. Then, we use u in the initial value problem for x, (2.18), and denote by $\Phi(x)$ the solution. Thus, $\Phi : H^2(0, T) \to H^2(0, T)$.

We show that the mapping Φ has a unique fixed point which is the unique solution of problem (2.16)–(2.19), when $d > 0$. To this end we need the folllowing estimate. Let w_i be the solutions of (A.6) and (A.7) for $i = 1, 2$. From (A.10), (A.6), and (A.7) we obtain

$$\frac{1}{2}\langle B\left(w_1 - w_2\right), \left(w_2 - w_2\right)\rangle(t) + \delta\int_0^t\ ||w_1 - w_2||_F^2\ \mathrm{d}s$$

$$\leq \lambda\int_0^t \langle B\left(w_1 - w_2\right), \left(w_2 - w_2\right)\rangle \mathrm{d}s + \int_0^t\ |x_1 - x_2|^2\ \mathrm{d}s. \tag{A.11}$$

By Gronwall's inequality, this implies the existence of a constant C, depending on T and λ but independent of w_i and x_i, such that

$$\langle B(w_1 - w_2), (w_1 - w_2)\rangle(t) + \delta \int_0^t ||w_1 - w_2||_F^2 \, \mathrm{d}s \le C \int_0^t |x_1 - x_2|^2 \, \mathrm{d}s.$$

In particular, since $w_i = (u_i, v_i)$, this shows that

$$||u_1(t) - u_2(t)||_V^2 \le C \int_0^t |x_1 - x_2|^2 \, \mathrm{d}s. \tag{A.12}$$

Now, from (2.18) we obtain

$$\Phi(x_1)(t) - \Phi(x_2)(t) = \int_0^t \int_0^s (R(\gamma_0 u_1(r), x_1(r)) - R(\gamma_0 u_2(r), x_2(r))) \, \mathrm{d}r \mathrm{d}s,$$

and it follows from (A.12) and Jensen's inequality that there is a constant C, independent of the x_i, such that

$$|\Phi(x_1)(t) - \Phi(x_2)(t)|^2 \le C \int_0^t \int_0^s \left(||u_1(r) - u_2(r)||_V^2 + |\Phi(x_1)(r) - \Phi(x_2)(r)|^2\right) \mathrm{d}r \mathrm{d}s$$

$$\le C \int_0^t \int_0^s \left(\int_0^r |x_1 - x_2|^2 \, d\tau + |\Phi(x_1)(r) - \Phi(x_2)(r)|^2\right) \mathrm{d}r \mathrm{d}s$$

$$\le C \int_0^t \int_0^s \int_0^r |x_1 - x_2|^2 \, d\tau \mathrm{d}r \mathrm{d}s + T \int_0^t |\Phi(x_1)(r) - \Phi(x_2)(r)|^2 \, \mathrm{d}r.$$

Consequently, using Gronwall's inequality once again yields the existence of a constant C, which depends on T but is independent of the x_i, such that

$$|\Phi(x_1)(t) - \Phi(x_2)(t)|^2 \le C \int_0^t \int_0^s \int_0^r |x_1 - x_2|^2 \, d\tau \mathrm{d}r \mathrm{d}s.$$

Iterating this inequality p times we find that there exists $\rho \in (0, 1)$ such that, for all p sufficiently large and $t \in [0, T]$,

$$\left|\Phi^p(x_1)(t) - \Phi^p(x_2)(t)\right|^2 \le \rho \, ||x_1 - x_2||_\infty^2 .$$

Thus, for sufficiently large p, the mapping Φ^p is a contraction on $C([0, T])$. Therefore, Φ has a unique fixed point x which, together with the corresponding u form the unique solution of the problem when $d > 0$. Now, since x solves the initial value problem (2.18), it follows that $x \in H^2(0, T)$. This establishes the existence and uniqueness part of Theorem 2.2 for $d > 0$.

To obtain the existence result when $d = 0$, we solve a sequence of problems with $d_n > 0$, with the associated sequence of solutions $\{u_{d_n}, x_{d_n}\}$, and pass to the limit $d_n \to 0$ as $n \to \infty$.

To simplify the notation, we temporarily omit the subscript n. Let $\{u_d, x_d\}$ be the solution of the problem, *i.e.*,

$$u_d, u_d' \in \mathcal{V}, \; u_d'' \in \mathcal{V}', \; x \in H^2(0, T), \tag{A.13}$$

$$u_d'' + Cu_d + dCu_d' + P(x_d, u_d) = f \quad \text{in } \mathcal{V}', \tag{A.14}$$

$$x_d''(t) + R(\gamma_0 u, x_d) = 0, \tag{A.15}$$

$$u_d'(0) = v_0 \in H, \; u_d(0) = u_0 \in V, \; x_d(0) = x_0, \; x_d'(0) = w_0, \tag{A.16}$$

$$u_d(0) = u_0, \; u_d'(0) = v_0 \in H. \tag{A.17}$$

We assume, as above, that $R(\gamma_0 u_0, x_0) = 0$.

We proceed to obtain estimates on $\{u_d\}$. We let the left-hand side in (A.14) act on u_d' and after integration of the result over $(0,t)$ we find

$$\frac{1}{2}\left|u_d'(t)\right|_H^2 - \frac{1}{2}|v_0|_H^2 + \frac{1}{2}||u_d(t)||_V^2 - \frac{1}{2}||u_0||_V + d\int_0^t \left|\left|u_d'(s)\right|\right|_V^2 \mathrm{d}s$$
$$+\int_0^t R(\gamma_0 u_d(s), x_d(s))\left(\gamma_0 u_d'(s) - x'(s)\right)\mathrm{d}s \tag{A.18}$$
$$\leq ||f||_{L^2(0,T;H)}^2 - \int_0^t R(\gamma_0 u_d(s), x_d(s))\, x_d'(s)\,\mathrm{d}s.$$

Since by assumption $R(\gamma_0 u_0, x_0) = 0$, the sixth term on the left-hand side is nonnegative and equals

$$\frac{\kappa}{2m}\left[(\gamma_0 u_d(t) - x_d(t) - g_2)_+^2 + (g_1 - \gamma_0 u_d(t) + x_d(t))_+^2\right],$$

while the last term is dominated by

$$C\int_0^t (||u_d(s)||_V + |x_d(s)|)\left|x_d'(s)\right|\mathrm{d}s \leq$$

$$C\int_0^t ||u_d(s)||_V^2\,\mathrm{d}s + C\int_0^t \left|x_d'(s)\right|^2\mathrm{d}s + \int_0^t |x_d(s)|^2\,\mathrm{d}s$$

$$\leq C\left(1 + \int_0^t ||u_d(s)||_V^2\,\mathrm{d}s + \int_0^t \left|x_d'(s)\right|^2\mathrm{d}s\right),$$

where C is independent of d. Therefore, it follows from (A.18) that there exists a constant C, independent of d, such that

$$\left|u_d'(t)\right|_H^2 + ||u_d(t)||_V^2 + d\int_0^t \left|\left|u_d'(s)\right|\right|_V^2\,\mathrm{d}s$$

$$\leq C\left(1 + \int_0^t ||u_d(s)||_V^2\,\mathrm{d}s + \int_0^t \left|x_d'(s)\right|^2\mathrm{d}s\right).$$

Application of Gronwall's inequality yields

$$\left|u_d'(t)\right|_H^2 + ||u_d(t)||_V^2 + d\int_0^t \left|\left|u_d'(s)\right|\right|_V^2\,\mathrm{d}s \leq C\left(1 + \int_0^t \left|x_d'(s)\right|^2\mathrm{d}s\right). \tag{A.19}$$

Next, we use the initial-value problem solved by x_d to estimate the term on the right-hand side. Multiplying both sides of (A.15) by x_d' and integrating over $(0,t)$, we obtain

$$\frac{1}{2}\left|x_d'(t)\right|^2 - \frac{1}{2}|x_0|^2 \leq \int_0^t |R(\gamma_0 u, x)|\left|x'(s)\right|\mathrm{d}s$$
$$\leq C\left(1 + \int_0^t ||u_d(s)||_V^2\,\mathrm{d}s + \int_0^t \left|x_d'(s)\right|^2\mathrm{d}s\right),$$

which, upon another application of Gronwall's inequality, leads to

$$\left|x_d'(t)\right|^2 \le C\left(1+\int_0^t ||u_d(s)||_V^2 \,\mathrm{d}s\right). \tag{A.20}$$

Using this in (A.19) yields

$$\left|u_d'(t)\right|_H^2 + ||u_d(t)||_V^2 + d\int_0^t \left|\left|u_d'(s)\right|\right|_V^2 \,\mathrm{d}s \le C\left(1+\int_0^t\int_0^s ||u_d(r)||_V^2 \,\mathrm{d}r\mathrm{d}s\right),$$

which, after additional application of Gronwall's inequality, implies

$$\left|u_d'(t)\right|_H^2 + ||u_d(t)||_V^2 + d\int_0^t \left|\left|u_d'(s)\right|\right|_V^2 \,\mathrm{d}s \le C,$$

where C is a constant independent of d. Now, from (A.20) and (2.18), we finally obtain the estimate,

$$\left|u_d'(t)\right|_H^2 + ||u_d(t)||_V^2 + d\int_0^t \left|\left|u_d'(s)\right|\right|_V^2 \,\mathrm{d}s + \left|x_d'(t)\right|^2 + \left|x_d''(t)\right|^2 + |x_d(t)|^2 \le C, \tag{A.21}$$

where C is independent of d.

We now restore the index n, and pass to the limit $d_n \to 0$ as $n \to \infty$. The estimate (A.21) and the inequality

$$d_n\left|\langle Cu_{d_n}', w\rangle\right| \le d_n^{1/2}\left|\left|u_{d_n}'\right|\right|_{L^2(0,T;V)} d_n^{1/2}\,||w||_{L^2(0,T;V)},$$

imply that

$$d_n C u_{d_n}' \to 0 \quad \text{strongly in } L^2(0,T;V). \tag{A.22}$$

Estimate (A.21) allows us to use a version of the Ascoli-Arzela theorem and conclude that, for a suitable subsequence (still indexed by n) in addition the following hold:

$$u_{d_n}' \to u' \text{ weak}^* \text{ in } L^\infty(0,T;H), \tag{A.23}$$

$$u_{d_n} \to u \text{ weak}^* \text{ in} L^\infty(0,T;V), \tag{A.24}$$

$$u_{d_n} \to u \text{ strongly in } C(0,T;W), \tag{A.25}$$

$$x_{d_n} \to x \text{ strongly in } C(0,T), \tag{A.26}$$

$$x_{d_n}' \to x' \text{ strongly in } C(0,T). \tag{A.27}$$

We now pass to the limit $d_n \to 0$ in (A.14) and (A.15) and conclude that the limit $\{u, x\}$ satisfies

$$u \in L^\infty(0,T;V),\ u' \in L^\infty(0,T;H),\ u'' \in L^2\left(0,T;V'\right), \tag{A.28}$$

$$u'' + Cu + P(x,u) = f \quad \text{in } L^2\left(0,T;V'\right), \tag{A.29}$$

$$x'' + \frac{\kappa}{m}\left[(u(0,t) - x(t) - g_2)_+ - (x(t) - g_1 - u(0,t))_+\right] = 0, \tag{A.30}$$

as well as the and initial conditions (2.19). This establishes the remaining part of Theorem 2.2.

References

1. K. L. Kuttler and M. Shillor, Vibrations of a beam between two stops. *Dyn. Contin. Discrete Impuls. Sys.* Ser B 8 (2001) 93–110.
2. Y. Dumont, Vibrations of a beam between stops: simulations and comparison of several numerical schemes. *in Math. Comput. in Simulation*, to appear .
3. Y. Dumont and M. Shillor, Simulations of beam's vibrations between stops. (1999) Preprint.
4. K. L. Kuttler, A. Park, M. Shillor and W. Zhang, Unilateral dynamic contact of two beams. *Math. Comput. Modelling* 34 (2001) 365–384.
5. A. Visintin, *Differential Models of Hysteresis*. Berlin: Springer-Verlag (1994) 407pp.
6. F. C. Moon and S. W. Shaw, Chaotic vibration of a beam with nonlinear boundary conditions. *J. Nonlin. Mech.* 18 (1983) 465–477.
7. F. C. Moon, *Chaotic and Fractal Dynamics*. New York: John Wiley & Sons (1992) 508pp.
8. M. Schatzman and M. Bercovier, Numerical approximation of a wave equation with unilateral constraints. *Math. Comput.* 53 (1989) 55–79.
9. L. Paoli and M. Schatzman, Resonance in impact problems. *Math. Comput. Modelling* 28 (1998) 385–406.
10. M. Schatzman, Uniqueness and continuous dependence on data for one-dimensional impact problems. *Math. Comput. Modelling* 28 (1998) 1–18.
11. G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics*. New-York: Springer-Verlag (1976) 397pp.
12. N. Kikuchi and J. T. Oden, *Contact Problems in Elasticity*. Philadelphia: SIAM (1988) 495pp.
13. W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity. Studies in Advanced Mathematics* **30**, Rhode Island: Americal Mathematical Society and International Press (2002) 442pp.
14. K. T. Andrews, M. Shillor and S. Wright, On the dynamic vibrations of an elastic beam in frictional contact with a rigid obstacle. *J. Elasticity* 42 (1996) 1–30.
15. A. Klarbring, A. Mikelic and M. Shillor, Frictional contact problems with normal compliance. *Int. J. Engng. Sci.* 26 (1988) 811–832.
16. J. A. C. Martins and J. T. Oden, A numerical analysis of a class of problems in elastodynamics with friction. *Comput. Meth. Appl. Mech. Engnr.* 40 (1983) 327–360.
17. K. T. Andrews, K. L. Kuttler and M. Shillor, On the dynamic behaviour of a thermoviscoelastic body in frictional contact with a rigid obstacle. *Euro. J. Appl. Math.* 8 (1997) 417–436.
18. K. L. Kuttler Time dependent implicit evolution equations. *Nonlinear Anal.* 10 (1986) 447–463.
19. K. L. Kuttler and M. Shillor Set-valued pseudomonotone maps and degenerate evolution inclusions. *Commun. Contemp. Math.* 1 (1999) 87–123.
20. M. Rochdi, M. Shillor and M. Sofonea, Quasistatic viscoelastic contact with normal compliance and friction. *J. Elasticity* 51 (1998) 105–126.
21. R. Adams, *Sobolev Spaces*. New York: Academic Press (1975) 268pp.
22. J. L. Lions and E. Magenes, *Nonhomogeneous Boundary Value Problems and Applications*. Vol. 1, Berlin: Springer (1972) 357pp.
23. W. Han, K. L. Kuttler, M. Shillor and M. Sofonea, Elastic beam in adhesive contact. *Int. J. Solids Structures* 39 (2002) 1145–1164.
24. H. M. Hilber, T. J. R. Hughes and R. L. Taylor, Improved numerical dissipation for time integration algorithms in structural dynamics. *Earthquake Engng. Struct. Dyn.* 5 (1977) 283–292.